Newton - Gregory backward interpolation formula.

To estimate the value of y for a given x which is near the end of the table or after the last value of the given range. (table)

$$f(x) = y_n + p\nabla y_n + \frac{p(p+1)}{2!}\nabla^2 y_n + \frac{p(p+1)(p+2)}{3!}\nabla^3 y_n + \cdots\cdots$$

where $x \to$ given value of x for which we need to find y.

$h = x_1 - x$, $\quad p = \dfrac{x - x_n}{h}$, $\quad x_n \to$ last value of x of the table.

1. Estimate $f(42)$ from the following data.

x	20	25	30	35	40	45		
$f(x)$	354	332	291	260	231			

given: $x = 42 \to$ which is near the end of the table

$\therefore$ we have to use NGBIF

. Construct backward difference table

	x	y	∇y	$\nabla^2 y$	$\nabla^3 y$	$\nabla^4 y$	$\nabla^5 y$
$x_0 \to$	20	354					
			-22				
$x_1 \to$	25	332		-19			
			-41		29		
	30	291		10		-37	
			-31		-8		45 $\nabla^5 y_n$
	35	260		2		8 $\nabla^4 y_n$	
			-29		0 $\nabla^3 y_n$		
	40	231		2 $\nabla^2 y_n$			
			-27 ∇y_n				
$x_n \to$	45	204 y_n					

$x = 42$ $\quad h = x_1 - x_0$

$x_n = 45$ $\quad = 25 - 20 = 5$

$$p = \frac{x - x_n}{h} = \frac{42-45}{5} = -\frac{3}{5} = -0.6$$

NGBIF :

$$f(x) = y_n + p\nabla y_n + \frac{p(p+1)}{2!}\nabla^2 y_n + \frac{p(p+1)(p+2)}{3!}\nabla^3 y_n + \frac{p(p+1)(p+2)(p+3)}{4!}\nabla^4 y_n + \frac{p(p+1)(p+2)(p+3)(p+4)}{5!}\nabla^5 y_n$$

$$= 204 + (-0.6)(-27) + \frac{(-0.6)\overset{0.4}{(-0.6+1)}}{2}(2) + \frac{(-0.6)\overset{+0.4}{(-0.6+1)}\overset{+1.4}{(-0.6+2)}}{6}0 + \frac{\overset{-0.2}{(-0.6)}\overset{0.4}{(-0.6+1)}\overset{1.4}{(-0.6+2)}\overset{2.4}{(-0.6+3)}}{\cancel{24}\ 3} 8 + \frac{(-0.6)\overset{0.4}{(-0.6+1)}\overset{1.4}{(-0.6+2)}\overset{2.4}{(-0.6+3)}\overset{3.4}{(-0.6+4)}}{\cancel{120}\ \cancel{24}\ 8} \cancel{48}$$

$= (204) + (16.2) - 0.24 + 0 - 0.2688 - 1.02816$

$= 218.663 = \underline{219}$

	x	θ	$\nabla\theta$	$\nabla^2\theta$	$\nabla^3\theta$	$\nabla^4\theta$	$\nabla^5\theta$
$x_0 \rightarrow$	40	184					
			20				
$x_1 \rightarrow$	50	204		2			
			22		0		
	60	226		2		0	0
			24		0		
	70	250		2		0	
			26		0		
	80	276		2			
			28				
$x_n \rightarrow$	90	304					

$x = 84, \quad x_n = 90, \quad h = x_1 - x_0$

$= 50 - 40$

$= 10$

$$p = \frac{x - x_n}{h}$$

$$= \frac{84 - 90}{10} = \frac{-6}{10} = -0.6$$

$$f(x) = y_n + p \nabla y_n + \frac{p(p+1)}{2!} \nabla^2 y_n + \frac{p(p+1)(p+2)}{3!} \nabla^3 y_n + \frac{p(p+1)(p+2)(p+3)}{4!} \nabla^4 y_n .$$

$$f(84) = 304 + (-0.6)\,28 + \frac{(-0.6)\overset{+0.4}{(-0.6+1)}}{2} \times 2 + \frac{(-0.6)\overset{0.4}{(-0.6+1)}\overset{1.4}{(-0.6+2)}}{6} \times 0$$

$f(84) = 286.96$ $\qquad$ $\therefore \boxed{f(84) = 286.96}$

3. find $f(105)$ from the following data

x	80	85	90	95	100
$f(x)$	5026	5674	6362	7088	7854.

8666

~~Ans:~~ 8

4. find $f(2.4)$ from the table

x	1.7	1.8	1.9	2.0	2.1	2.2	2.3
$y = e^x$	5.474	6.050	6.686	7.389	8.166	9.025	9.974

5. Estimate the population for the year 1986 from the Table

year x	195	196	1971	1981	1991
Population	46	66	81	93	101

Numerical analysis

Factorial notation:

A factorial polynomial $x^{(n)}$ is defined as.

$$x^{(n)} = x(x-h)(x-2h)\ldots(x-(n-1)h)$$

where n is a positive integer and h is the interval of difference.

ie $x^{(1)} = x$

$x^{(2)} = x(x-1)$

$x^{(3)} = x(x-1)(x-2)$

$x^{(4)} = x(x-1)(x-2)(x-3)$.

so on..

Problems: 1. Express $f(x) = 3x^3 + x^2 + x + 1$ as a factorial polynomial. take $h=1$.

Given: $f(x) = 3x^3 + x^2 + x + 1$.

Let $g(x) = A\,x^{(3)} + B\,x^{(2)} + C\,x^{(1)} + D$ —— ①.

be the factorial notation of $f(x)$.

0	3	1	1	1
	0	+0	+0	+0
				$\boxed{1}$ → D
1	3	1	1	
	0	+3	+4	
			$\boxed{5}$ → C	
2	3	4		
	0	6		
		$\boxed{10}$ → B		
3	3			
	0			
	3 → A			

Sub in ①.

$$g(x) = 3\,x^{(3)} + 10\,x^{(2)} + 5\,x^{(1)} + 1$$

2. Express the following polynomial and their successive differences in factorial notation.

a (a) $3x^3 - 4x^2 + 3x - 11$.

Let $f(x) = 3x^3 - 4x^2 + 3x - 11$

I Let $g(x) = A\,x^{(3)} + B\,x^{(2)} + C\,x^{(1)} + D$

$$
\begin{array}{c|cccc}
0 & 3 & -4 & 3 & -11 \\
 & 0 & 0 & 0 & 0 \\
\hline
1 & 3 & -4 & 3 & \boxed{-11} \rightarrow D \\
 & 0 & +3 & -1 & \\
\hline
2 & 3 & -1 & \boxed{2} - C & \\
 & 0 & 6 & & \\
\hline
3 & 3 & \boxed{5} \rightarrow B & & \\
 & 0 & & & \\
\hline
 & 3 \rightarrow A & & &
\end{array}
$$

Def^n

$\Delta x^{(3)} = 3x^{(2)}$

$\Delta x^{(10)} = 10x^{(9)}$

$\Delta(11) = 0$

$g(x) = 3x^{(3)} + 5x^{(2)} + 2x^{(1)} - 11$

II $\Delta g(x) = 9x^{(2)} + 10x^{(1)} + 2$

$\Delta^2 g(x) = \Delta\left(9x^{(2)} + 10x^{(1)} + 2\right)$

$= 18x^{(1)} + 10$

$\Delta^3 g(x) = \Delta\left(18x^{(1)} + 10\right)$

$= 18 + 0 = 18.$

$x^4 - 12x^3 + 42x^2 - 30x + 9$

(c) $3x^3 - 4x^2 + 8x + 2$

d) $2x^4 - 7x^{(2)} + 5x - 13$

$0 \mid 2,\ 0,\ -7,\ 5,\ -13$

e. $x^4 - 5x^3 + 3x + 4$.

3. Obtain a function whose first difference is
$6x^2 + 10x + 11$

Let the function be $f(x)$.

I then $\Delta f(x) = 6x^2 + 10x + 11$ (given)

$= A x^{(2)} + B x^{(1)} + C$

$= A x^{(2)} + B x^{(1)} + C$

$$\begin{array}{c|ccc}
0 & 6 & 10 & 11 \\
 & 0 & 0 & 0 \\
\hline
1 & 6 & 10 & \boxed{11} \\
 & 0 & 6 & \\
\hline
2 & 6 & \boxed{16} \rightarrow B & \\
 & 0 & & \\
\hline
 & 6 \rightarrow A & &
\end{array}$$

$\Delta f(x) = 6x^{(2)} + 16x^{(1)} + 11$

II $f(x) = \frac{1}{\Delta}\left(6x^{(2)} + 16x^{(1)} + 11\right)$

$f(x) = \cancel{6}^{2} \cdot \frac{x^{(3)}}{\cancel{3}} + \cancel{16}^{8} \frac{x^{(2)}}{\cancel{2}} + 11x^{(1)} + k$

$f(x) = 2x^{(3)} + 8x^{(2)} + 11x^{(1)} + k$.

$\frac{1}{\Delta} x^{(5)} = \frac{x^{(6)}}{6}$ like integration

$\frac{1}{\Delta} x^{(10)} = \frac{x^{(11)}}{11}$

III $f(x) = 2x(x-1)(x-2) + 8(x)(x-1) + 11x + k$

$f(x) = 2x^3 + 2x^2 + 7x + k$.

required polynomial.

Problems

2. obtain function whose first difference is

$$x^4 - 5x^3 + 3x + 4$$

given. $\Delta f(x) = x^4 - 5x^3 + 3x + 4$

let $\Delta f(x) = A x^{(4)} + B x^{(3)} + C x^{(2)} + D x^{(1)} + E$

$$\begin{array}{c|ccccc}
0 & 1 & -5 & 0 & 3 & 4 \\
 & 0 & 0 & 0 & 0 & 0 \\
\hline
1 & 1 & -5 & 0 & 3 & \boxed{4} \to E \\
 & 0 & 1 & -4 & -4 & \\
\hline
2 & 1 & -4 & -4 & \boxed{-1} \to D & \\
 & 0 & 2 & -4 & & \\
\hline
3 & 1 & -2 & \boxed{-8} \to C & & \\
 & 0 & 3 & & & \\
\hline
4 & 1 & \boxed{+1} \to B & & & \\
 & 0 & & & & \\
\hline
 & 1 \to A & & & &
\end{array}$$

$\therefore \Delta f(x) = 1\, x^{(4)} + 1\, x^{(3)} - 8\, x^{(2)} - 1\, x^{(1)} + 4$

$$f(x) = \frac{1}{\Delta}\left[x^{(4)} + x^{(3)} - 8\, x^{(2)} - x^{(1)} + 4\right]$$

$$= \frac{x^{(5)}}{5} + \frac{x^{(4)}}{4} - 8\,\frac{x^{(3)}}{3} - \frac{x^{(2)}}{2} + 4x^{(1)} + k.$$

$$= \frac{1}{5} x(x-1)(x-2)(x-3)(x-4) + \frac{1}{4} x(x-1)(x-2)(x-3) - 8\, x(x-1)(x-2) - \frac{1}{2} x(x-1)$$

3. obtain the function whose first difference given

a). $3x^2 + 9x + 4$

(c) $x^3 + 4x^2 + 9x + 12$

b) $2x^3 - 6x^2 + 7x + 10$

(d) $2x^3 + 5x^2 - 6x + 13$

Proof by Separation of symbols :

Some basic concepts :

- Relation between operators $\Delta, \nabla, E, E^{-1}$

1. $\boxed{\Delta = E-1}$, $\boxed{E = 1+\Delta}$

2. $\boxed{\nabla = 1-E^{-1}}$, $\boxed{E^{-1} = 1-\nabla}$, $\boxed{E = (1-\nabla)^{-1}}$

- Definitions : * $\Delta y_0 = y_1 - y_0$

lly $\Delta u_0 = u_1 - u_0$

$\Delta u_1 = u_2 - u_1$

* $EU_0 = u_1$

OR

* $u_1 = EU_0, \quad U_2 = E^2 u_0, \quad U_3 = E^3 U_0 \dots\dots$

* $U_{x-1} = E^{-1} U_x, \; U_{x-2} = E^{2} U_x, \; U_{x-3} = E^{3} U_x \dots\dots$

- Formulas :

* $(1+x)^{-1} = 1 - x + x^2 - x^3 + x^4 \dots\dots$ } Binomial Expansion.

* $e^x = 1 + \frac{x}{1!} + \frac{x^2}{2!} + \frac{x^3}{3!} + \dots\dots$

Ex: $e^{xE} = 1 + \frac{xE}{1!} + \frac{x^2E^2}{2!} + \frac{x^3E^3}{3!} + \dots\dots$

$\left(1+\frac{\Delta}{2}\right)^{-1} = 1 - \frac{\Delta}{2} + \frac{\Delta^2}{2^2} - \frac{\Delta^3}{2^3} + \dots\dots$

* $(1+x)^n = 1 + {}^nC_1 x + {}^nC_2 x^2 + {}^nC_3 x^3 + \dots + x^n$

* $(1-x)^{-1} = 1+x+x^2+x^3+\cdots$

Ex $\left(1-\frac{x\Delta}{(1-x)}\right)^{-1} = 1+\frac{x\Delta}{(1-x)}+\frac{x^2\Delta^2}{(1-x)^2}+\cdots$

* Geometric progression

$a+ar+ar^2+ar^3+\cdots+ar^{n-1} = S$

Then $S = \frac{a(r^n-1)}{(r-1)}$ $\quad |r|>1$

$S = \frac{a(1-r^n)}{(1-r)}$ $\quad |r|<1.$

$a+ar+ar^2+ar^3+\cdots+ar^{n-1}+ar^n = S$

$S = a\left(\frac{r^{n+1}-1}{r-1}\right)$

$a+ar+ar^2+ar^3+\cdots\cdots\infty$

$S = \frac{a}{1-r}$

Ex: $1+\frac{\Delta}{E}+\frac{\Delta^2}{E^2}+\cdots+\frac{\Delta^{n-1}}{E^{n-1}} = \frac{1\left(\left(\frac{\Delta}{E}\right)^n-1\right)}{\left(\frac{\Delta}{E}-1\right)}$

$\frac{x\Delta}{(1+x)}-\frac{x^2\Delta^2}{(1+x)^2}+\frac{x^3\Delta^3}{(1+x)^3}-\cdots = \frac{\frac{\Delta x}{(1+x)}}{1+\frac{\Delta x}{(1+x)}}$

Problems: 1. Use the method of separation of symbols to prove $u_0 - u_1 + u_2 - \cdots = \frac{1}{2}u_0 - \frac{1}{4}\Delta u_0 + \frac{1}{8}\Delta^2 u_0 - \cdots$

Proof:

$$L.H.S = u_0 - u_1 + u_2 - u_3 + u_4 - \cdots$$

$$= u_0 - Eu_0 + E^2 u_0 - E^3 u_0 + \cdots$$

$$= (1 - E + E^2 - E^3 + \cdots) u_0$$

$$= (1+E)^{-1} u_0$$

$u_1 = Eu_0$
$u_2 = E^2 u_0$
$u_3 = E^3 u_0$

$$E = 1 + \Delta.$$

$$= (1 + 1 + \Delta)^{-1} u_0 = (2+\Delta)^{-1} u_0$$

$$= \left[2\left(1 + \frac{\Delta}{2}\right)\right]^{-1} u_0 = \frac{1}{2} \cdot \left(1 + \frac{\Delta}{2}\right)^{-1} u_0$$

$$= \frac{1}{2}\left[1 - \frac{\Delta}{2} + \frac{\Delta^2}{2^2} - \frac{\Delta^3}{2^3} + \cdots\right] u_0$$

$$= \frac{1}{2}u_0 - \frac{\Delta}{4} u_0 + \frac{\Delta^2}{8} u_0 - \cdots = R.H.S.$$

2. Prove that.

$$u_0 + u_1 + u_2 + u_3 + \cdots + u_n = {}^{(n+1)}C_1 u_0 + {}^{(n+1)}C_2 \Delta u_0 + \cdots + \Delta^n u_0$$

using separation of symbols.

$$L.H.S = U_0 + U_1 + U_2 + U_3 + \cdots + U_n.$$

$$= U_0 + EU_0 + E^2 U_0 + E^3 U_0 + \cdots + E^n U_0$$

$$= (1 + E + E^2 + E^3 + \cdots + E^n) U_0$$

$$= \left(\frac{E^{n+1} - 1}{E - 1}\right) U_0$$

Put $E = 1 + \Delta$.

$$= \left[\frac{(1+\Delta)^{n+1} - 1}{\Delta}\right] U_0 = \frac{1}{\Delta}\left[(1+\Delta)^{n+1} - 1\right] U_0$$

From Binomial Expansion,

$$(1+\Delta)^{n+1} = 1 + {}^{(n+1)}C_1 \Delta + {}^{(n+1)}C_2 \Delta^2 + \cdots + \Delta^{n+1}$$

$\therefore$

$$L.H.S = \frac{1}{\Delta}\left[\cancel{1} + {}^{(n+1)}C_1 \Delta + {}^{(n+1)}C_2 \Delta^2 + \cdots + \Delta^{n+1} \cancel{-1}\right] U_0$$

$$= \left[{}^{(n+1)}C_1 \frac{\cancel{\Delta}}{\cancel{\Delta}} + {}^{(n+1)}C_2 \frac{\Delta^{\cancel{2}}}{\cancel{\Delta}} + {}^{(n+1)}C_3 \frac{\Delta^{\cancel{3}\,2}}{\cancel{\Delta}} + \cdots + \frac{\Delta^{n+1}}{\Delta}\right] U_0$$

$$= \left[{}^{(n+1)}C_1 + {}^{(n+1)}C_2 \Delta + {}^{(n+1)}C_3 \Delta^2 + \cdots + \Delta^n\right] U_0$$

$$= {}^{(n+1)}C_1 U_0 + {}^{(n+1)}C_2 \Delta U_0 + {}^{(n+1)}C_3 \Delta^2 U_0 + \cdots + \Delta^n U_0$$

$= R.H.S.$

— o —

use separation of symbols to prove.

$$U_x = U_{x-1} + \Delta U_{x-2} + \Delta^2 U_{x-3} + \cdots + \Delta^{n-1} U_{x-n} + \Delta^n U_{x-n}$$

Put $U_{x-1} = E^{-1} U_x$

$U_{x-2} = E^{-2} U_x$

$\vdots$

$U_{x-n} = E^{-n} U_x$.

$$U_x = E^{-1} U_x + \Delta E^{-2} U_x + \Delta^2 E^{-3} U_x + \cdots + \Delta^{n-1} E^{-n} U_x + \Delta^n E^{-n} U_x$$

$$R.H.S = \left(E^{-1} + \Delta E^{-2} + \Delta^2 E^{-3} + \Delta^3 E^{-4} + \cdots + \Delta^{n-1} E^{-n}\right) U_x + \Delta^n E^{-n} U_x$$

$$= E^{-1}\left(1 + \Delta E^{-1} + \Delta^2 E^{-2} + \cdots + \Delta^{n-1} E^{-n+1}\right) U_x + \Delta^n E^{-n} U_x$$

$$= E^{-1}\left(1 + \frac{\Delta}{E} + \frac{\Delta^2}{E^2} + \cdots + \frac{\Delta^{n-1}}{E^{n-1}}\right) U_x + \frac{\Delta^n}{E^n} U_x$$

$$\boxed{1 + x + x^2 + x^3 + \cdots + x^n = 1\frac{(x^n - 1)}{x-1}}$$

$$= E^{-1}\left[\frac{\left(\frac{\Delta}{E}\right)^n - 1}{\frac{\Delta}{E} - 1}\right] U_x + \frac{\Delta^n}{E^n} U_x$$

$$= \frac{1}{\cancel{E}}\left[\left(\frac{\Delta^n - E^n}{E^n}\right)\left(\frac{\cancel{E}}{\Delta - E}\right)\right] U_x + \frac{\Delta^n}{E^n} U_x .$$

$$\boxed{\Delta - E = -1 \quad \because 1 + \Delta = E}$$

$$= \left[\left(\frac{\Delta^n - E^n}{E^n}\right)(-1)\right] U_x + \frac{\Delta^n}{E^n} U_x .$$

$$= -\cancel{\frac{\Delta^n}{E^n}} U_x + \cancel{\frac{E^n}{E^n}} U_x + \cancel{\frac{\Delta^n}{E^n}} U_x = U_x = L.H.S.$$

By separation of symbols prove that:

$$u_0 + u_1 x + u_2 x^2 + \dots + \dots \text{ to } \infty =$$

$$\frac{u_0}{(1-x)} + \frac{x \Delta u_0}{(1-x)^2} + \frac{x^2 \Delta^2 u_0}{(1-x)^3} + \dots \text{ to } \infty$$

$$L.H.S = u_0 + u_1 x + u_2 x^2 + \dots + \text{ to } \infty$$

$$= u_0 + E u_0 x + E^2 u_0 x^2 + \dots \infty$$

$$= (1 + Ex + E^2 x^2 + \dots) u_0$$

$$\left\{ a + ar + ar^2 + \dots \infty = \frac{a}{1-r} \right\}$$

$$= \left(\frac{1}{1-xE}\right) u_0$$

$$= \left[\frac{1}{1-x(1+\Delta)}\right] u_0 \qquad \because E\ (1+\Delta)$$

$$= \left(\frac{1}{(1-x-x\Delta)}\right) u_0 = \frac{1}{(1-x)} \left[\frac{1}{1-\frac{x\Delta}{1-x}}\right] u_0$$

$$= \frac{1}{(1-x)} \left[1 - \frac{x\Delta}{1-x}\right]^{-1} u_0$$

$$= \left(\frac{1}{1-x}\right) \left[1 - \frac{x\Delta}{1-x} + \frac{x^2\Delta^2}{(1-x)^2} + \frac{x^3\Delta^3}{(1-x)^3} + \dots\right] u_0$$

$$= \left[\frac{1}{(1-x)} - \frac{x\Delta}{(1-x)^2} + \frac{x^2\Delta^2}{(1-x)^3} + \dots\right] u_0$$

$$= \frac{u_0}{(1-x)} - \frac{x\Delta u_0}{(1-x)^2} + \frac{x^2}{(1-x)^3} \Delta^2 u_0 + \dots$$

$$= R.H.S.$$

— By separation of symbols prove that

$$u_0 + \frac{u_1 x}{1!} + \frac{u_2 x^2}{2!} + \cdots = e^x\left[u_0 + \frac{x \Delta u_0}{1!} + \frac{x^2 \Delta^2 u_0}{2!} + \cdots\right]$$

$$L.H.S = u_0 + \frac{u_1 x}{1!} + \frac{u_2 x^2}{2!} + \cdots$$

$$= u_0 + \frac{E u_0 x}{1!} + \frac{E^2 u_0 x^2}{2!} + \cdots$$

$$= \left[1 + \frac{Ex}{1!} + \frac{E^2 x^2}{2!} + \frac{E^3 x^3}{3!} + \cdots\right] u_0$$

$$= e^{Ex} u_0$$

$$= e^{(1+\Delta)x} u_0$$

$$= e^x . e^{\Delta x} u_0$$

$$= e^x\left[1 + \frac{x\Delta}{1!} + \frac{x\Delta^2}{2!} + \frac{x\Delta^3}{3!} + \cdots\right] u_0$$

$$= e^x\left[u_0 + \frac{x \Delta u_0}{1!} + \frac{x}{2!}\Delta^2 u_0 + \frac{x}{3!}\Delta^3 u_0 + \cdots\right]$$

$$= R.H.S.$$

6. Prove by separation of symbols.

$$(u_1-u_0) - x(u_2-u_1) + x^2(u_3-u_2) - \cdots$$

$$= \frac{\Delta u_0}{1+x} - \frac{x\Delta^2 u_0}{(1+x)^2} + \frac{x^2\Delta^3 u_0}{(1+x)^3} - \cdots$$

$$L.H.S = (u_1-u_0) - x(u_2-u_1) + x^2(u_3-u_2) - \cdots$$

$u_1 - u_0 = \Delta u_0$

$u_2 - u_1 = \Delta u_1$ By def^n

$u_3 - u_2 = \Delta u_2$

$$= \Delta u_0 - x\Delta u_1 + x^2 \Delta u_2 - \cdots$$

$$= \Delta u_0 - x\Delta E u_0 + x^2 \Delta E^2 u_0 - x^3 \Delta E^3 u_0 + \cdots$$

$$= [1 - xE + x^2E^2 - x^3E^3 + \cdots]\Delta u_0$$

$$= (1+xE)^{-1}\Delta u_0 \quad \text{—— (1)}$$

$$R.H.S = \frac{\Delta u_0}{1+x} - \frac{x\Delta^2 u_0}{(1+x)^2} + \frac{x^2\Delta^3 u_0}{(1+x)^3} - \cdots$$

multiply and divide by x.

$$= \frac{1}{x}\left[\frac{x\Delta u_0}{1+x} - \frac{x^2\Delta^2 u_0}{(1+x)^2} + \frac{x^3\Delta^3 u_0}{(1+x)^3} - \cdots\right]$$

$$= \frac{1}{x}\left[\frac{x\Delta}{1+x} - \frac{x^2\Delta^2}{(1+x)^2} + \frac{x^3\Delta^3}{(1+x)^3} - \cdots\right]u_0$$

$a = \frac{x\Delta}{1+x}$

$r = -\left(\frac{x\Delta}{1+x}\right)$

$S_\infty = \frac{a}{1-r}$

$$= \frac{1}{x}\left[\frac{\frac{x\Delta}{1+x}}{1-\left(-\frac{x\Delta}{1+x}\right)}\right]u_0$$

$$= \frac{1}{x}\left[\frac{x\Delta/(1+x)}{1+\frac{x\Delta}{1+x}}\right]u_0 = \frac{1}{\cancel{x}}\left[\frac{\frac{\cancel{x}\Delta}{\cancel{1+x}}}{\frac{1+x+x\Delta}{\cancel{1+x}}}\right]u_0 = \left(\frac{\Delta}{1+x+x\Delta}\right)u_0$$

$$= \left(\frac{\Delta}{1+x(1+\Delta)}\right)u_0$$

$$= \left(\frac{\Delta}{1+xE}\right)u_0$$

$$= (1+xE)^{-1}\Delta u_0 \quad \text{—— (2)}$$

(1) & (2) $\Rightarrow$

L.H.S = R.H.S

7. P.T.

$$u_x - {}^nc_1 u_{x-1} + {}^nc_2 u_{x-2} - {}^nc_3 u_{x-3} + \cdots = \Delta^n u_{x-n}$$

$$\begin{aligned}
L.H.S &= u_x - {}^nc_1 u_{x-1} + {}^nc_2 u_{x-2} - {}^nc_3 u_{x-3} + \cdots \\
&= u_x - {}^nc_1 E^{-1} u_x + {}^nc_2 E^{-2} u_x - {}^nc_3 E^{-3} u_x + \cdots \\
&= (1 - {}^nc_1 E^{-1} + {}^nc_2 E^{-2} - {}^nc_3 E^{-3} + \cdots) u_x \\
&= (1 - E^{-1})^n u_x \\
&= \left(1 - \frac{1}{E}\right)^n u_x \\
&= \frac{(E-1)^n}{E^n} u_x = \frac{\Delta^n}{E^n} u_x \\
&= \Delta^n E^{-n} u_x \\
&= \Delta^n u_{x-n} = R.H.S.
\end{aligned}$$